

Some Comments on the Holographic Heavy Quark Potential in a Thermal Bath

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The heavy quark potential of a thermal Yang-Mills theory in strong coupling limit is explored in terms of the holographic principle. With a fairly general AdS/QCD metric the heavy quark potential displays a kink-like screening in the plasma phase. This behavior may conflict the causality of a field theory that is mathematically equivalent to the thermal Yang-Mills.

1. INTRODUCTION

The heavy-quark potential (the interaction energy between a quark and its antiparticle in the infinite mass limit) is a very important quantity of QCD. Not only does it provide the information of the quarkonium dissociation which signals the formation of QGP in heavy ion collisions[1], but also is one of the basic probes to explore the phase diagram with nonzero temperature and baryon density. While the potential is Coulomb like for a short separation, $r \ll \Lambda_{\text{QCD}}^{-1}$, the potential rises linearly at large distance, $r \gg \Lambda_{\text{QCD}}^{-1}$ [2] in the confined phase without light quarks as is suggested by the Regge behavior of meson spectra and is expected to be screened within a radius of the order T^{-1} in the deconfined phase.

Mathematically, the heavy-quark potential can be extracted from the expectation value of a Wilson loop, or the correlator between two Polyakov loops.[3] The highly nonperturbative nature of infrared QCD makes it analytically intractable, especially in the confined phase. The lattice simulation of the Wilson loops and/or the Polyakov loops has accumulated sufficient evidence supporting the linear potential below the deconfinement temperature and the screened potential in the plasma phase[4–12]. Furthermore a resummation of perturbative series yields an exponentially screening at high temperature in weak coupling [13, 14].

The advent of the holographic principle [15, 16], especially the AdS/CFT correspondence[17] opens a new avenue towards analytic treatments of the strong coupling limit of a gauge theory, in particular, the $N = 4$ supersymmetric $SU(N_c)$ Yang-Mills theory at large N_c and large 't Hooft coupling, $\lambda \equiv N_c g_{\text{YM}}^2$ with g_{YM} the Yang-Mills coupling. The heavy-quark potential[17–30] at zero temperature is Coulomb like with the strength proportional to $\sqrt{\lambda}$. [17, 20, 21] At a nonzero temperature, [18, 19, 22] the potential displays a kink-like screening with a radius of $r_s \simeq 0.754(\pi T)^{-1}$, i.e. the potential is flattened out beyond r_s . This transition is interpreted as string melting in [18] and will be referred to as the kink-like screening in this paper. But the super Yang-Mills is not QCD and the conformal property of the former makes the Coulomb like behavior the only possible outcome at zero temperature following a dimensional argument. A cousin of AdS/CFT with an infrared cutoff, AdS/QCD, has been actively investigated and is able to provide a linear heavy quark-potential at zero temperature. The nonzero temperature behavior were found to fit lattice data quite well.[23–25]

In this paper, we shall explore analytically the screening property of the heavy-quark potential within the framework of AdS/QCD. In the next section, the heavy-quark potential of the $N = 4$ super Yang-Mills at a nonzero temperature following AdS/CFT correspondence will be reviewed, where we shall also set up our notations. In the section III, we shall show that under a fairly general conditions of the metric underlying AdS/QCD, the screening remains kink-like, like that of the super Yang-Mills. In other words, AdS/QCD cannot provide a exponentially screening potential in the plasma phase. Different scenarios of a smooth screening behaviors proposed in the literature will be discussed in the section IV, where we shall also point out a potential gap between the kink-like screening potential and the fundamental principles of quantum field theories.

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2. THE HEAVY-QUARK POTENTIAL IN A $N = 4$ SUPER YANG-MILLS PLASMA IN STRONG COUPLING

AdS/CFT correspondence relates the type IIB superstring theory in $AdS_5 \times S^5$ background to the $N = 4$ supersymmetric $SU(N_c)$ Yang-Mills theory at the AdS boundary. In particular, the thermal expectation value of a Wilson loop C , $\langle W[C] \rangle$, at large N_c and large 't Hooft coupling constant $\lambda \equiv N_c g_{\text{YM}}^2$ corresponds to the minimum area of a string world sheet in the Schwarzschild- AdS_5 metric,

$$ds^2 = \frac{1}{z^2} \left[\left(1 - \frac{z^4}{z_h^4} \right) d\tau^2 + \left(1 - \frac{z^4}{z_h^4} \right)^{-1} dz^2 + d\vec{x}^2 \right] \quad (2.1)$$

bounded by the loop C at the boundary $z = 0$, i.e.

$$\langle W[C] \rangle = \text{const.} e^{-\min.(S_{\text{NB}}[C])} \quad (2.2)$$

where the Nambu-Goto action

$$S_{\text{NB}}[C] = \frac{\mathcal{S}}{2\pi\alpha} = \frac{1}{2\pi\alpha} \int d^2\sigma \sqrt{g} \quad (2.3)$$

with \mathcal{S} the world sheet area, and g the determinant of the induced world sheet metric, $ds^2 = g_{ab} d\sigma^a d\sigma^b$ ($a, b = 0, 1$). The horizon radius z_h corresponds to the temperature $T = (\pi z_h)^{-1}$ and the string tension corresponds to the 't Hooft coupling $\sqrt{\lambda} = \alpha^{-1}$.

The Wilson loop for the free energy of a single heavy quark or antiquark is a Polyakov line winding up the Euclidean time dimension and will be denoted by \mathcal{C}_1 . The Wilson loop for the free energy of a pair of heavy quark and antiquark consists of two Polyakov lines running in opposite directions and will be denoted by \mathcal{C}_2 . The heavy quark potential $V(r)$ corresponds to the interaction part of the free energy of the quark pair and is extracted from the ratio

$$\frac{\langle W[\mathcal{C}_2] \rangle}{|\langle W[\mathcal{C}_1] \rangle|^2} = \frac{\text{Tr} \langle \mathcal{P}(\vec{r}) \mathcal{P}^\dagger(0) \rangle}{|\text{Tr} \langle \mathcal{P}(0) \rangle|^2} \equiv e^{-\frac{V(r)}{T}} \quad (2.4)$$

where the Wilson line operator ¹

$$\mathcal{P}(\vec{r}) = \mathcal{T} e^{-i \int_0^{T^{-1}} dt A_0(\vec{r}, \tau)} \quad (2.5)$$

with $A_0(\vec{r}, \tau)$ the temporal component of the Lie Algebra valued gauge potential and \mathcal{T} the time ordering operator. The solution of the Euler-Lagrange equation for \mathcal{C}_1 is a world sheet with constant 3D spatial coordinates \vec{x} extending from the AdS boundary $z = 0$ to the black hole horizon $z = z_h$ and its NG action will be denoted by $\frac{1}{T} \mathcal{A}_1$. The solution of the Euler-Lagrange equation for \mathcal{C}_2 can be either a connected nontrivial world sheet shown in Fig.1a or two parallel world sheets shown in Fig.1b with each identical to the world sheet of \mathcal{C}_1 . The NG action of the former will be denoted by $\frac{1}{T} \mathcal{A}_2$ while that of the latter is given by $\frac{2}{T} \mathcal{A}_1$ and the minimum of them contributes to the free energy. It follows from (2.4) that

$$V(r) = \min(\mathcal{A}, 0). \quad (2.6)$$

where $\mathcal{A} \equiv \mathcal{A}_2 - 2\mathcal{A}_1$ and this combination cancels the UV divergences pertaining to \mathcal{A}_1 and \mathcal{A}_2 . We shall name $\mathcal{A}(r)$ as the candidate potential.

For $N = 4$ super Yang-Mills, the Euler-Lagrange equation of minimizing $S_{\text{NB}}[C]$ yields the following parametric form of the function candidate potential $\mathcal{A}(r)$ [17–19]

$$\begin{cases} r = 2\sqrt{z_h^4 - z_c^4} \int_0^{z_c} dz \frac{z^2}{\sqrt{(z_h^4 - z^4)(z_c^4 - z^4)}} \\ \mathcal{A} = \frac{\sqrt{\lambda z_c^2}}{\pi z_h^2} \left[\int_0^{z_c} \frac{dz}{z^2} \left(\sqrt{\frac{z_h^4 - z^4}{z_c^4 - z^4}} - 1 \right) - \int_0^{z_h} dz \frac{1}{z^2} \right] \end{cases} \quad (2.7)$$

¹ Strictly speaking, the "heavy quark" in the super Yang-Mills refers to the heavy W-boson of the symmetric breaking from $SU(N_c)$ to $SU(N_c - 1)$ and the Polyakov line operator contains the contribution from the scalar field. See e.g. [17] for details.

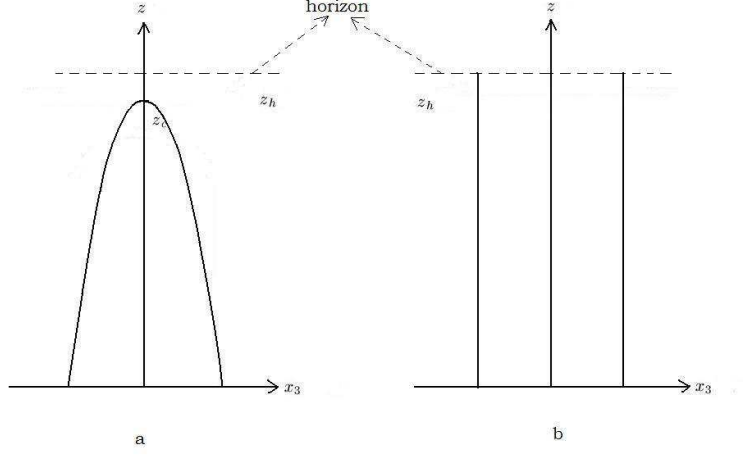


FIG. 1: (a) The connected nontrivial world sheet from the boundary $z = 0$ extending to $z_c < z_h$. (b) is two parallel world sheets starting from the boundary $z = 0$ and ending at the horizon $z = z_h$. The upper dashed lines represent the black hole horizon.

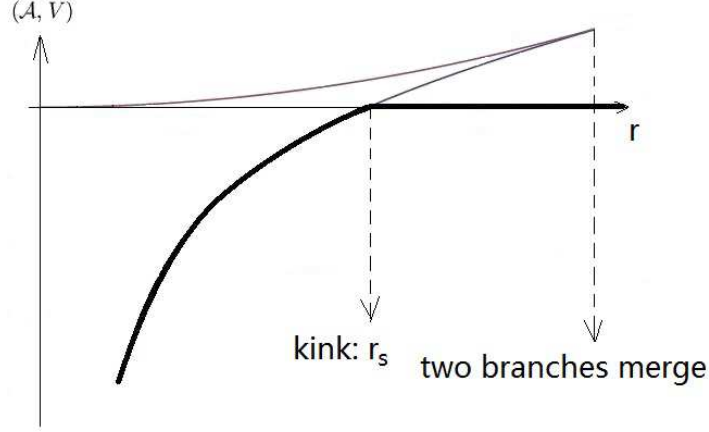


FIG. 2: The candidate potential (thin line) and the heavy quark potential (thick line) of the super Yang-Mills. Both coincide below the r -axis.

where the parameter z_c is the maximum extension of the world sheet in the bulk. The candidate potential \mathcal{A} as a function of the distance, shown in Fig.2, consists of two branches. As z_c starts from the AdS boundary, \mathcal{A} starts with an attractive Coulomb like form

$$\mathcal{A} \simeq -\frac{4\pi^2\sqrt{\lambda}}{\Gamma^4(1/4)r} \quad (2.8)$$

for $rT \ll 1$ along the lower branch. Then \mathcal{A} becomes positive at $r = 0.745/(\pi T) \equiv r_s$ and reaches the end of the lower branch which corresponds to the maximum of r as a function of z_c . Beyond this value of z_c , the potential follows the upper (repulsive) branch and decreases to zero as $z_c \rightarrow z_h$ ($r \rightarrow 0$). According to (2.6), the potential $V(r)$ is given by \mathcal{A} for $r < r_s$ and vanishes beyond r_s . Numerically, this screening potential can be well approximated by a truncated Coulomb potential

$$V(r) = \begin{cases} \kappa \left(\frac{1}{r} - \frac{1}{a} \right) & \text{for } r \leq a \\ 0 & \text{for } r > a \end{cases} \quad (2.9)$$

with $\kappa = -\frac{4\pi^2\sqrt{\lambda}}{\Gamma^4(1/4)}$ and $a = \frac{4\pi^2}{\Gamma^4(1/4)T} \simeq \frac{0.736}{\pi T}$. [31].

3. THE HEAVY-QUARK POTENTIAL WITH A GENERAL ADS/QCD METRIC AT A NONZERO TEMPERATURE

Unlike the $N = 4$ super Yang-Mills, the real QCD is characterized by an intrinsic energy scale, Λ_{QCD} , and the metric of its gravity dual, if exists, should carry a length scale other than the horizon radius. The most general metric of AdS/QCD take the form

$$ds^2 = \frac{w(z)}{z^2} [f(z)d\tau^2 + \frac{1}{f(z)}dz^2 + d\vec{x}^2] \quad (3.1)$$

different proposals for the expressions of the warp factor $w(z)$ and the function $f(z)$ have been explored in the literature. They may be simply specified to warrant an analytic treatment[32–34] or may be dictated by the solution of Einstein equation coupled to a dilaton field [35]. A number of general conditions should be satisfied by $w(z)$ and $f(z)$: 1) The conformal invariance of QCD in UV limit requires that $w(0) = f(0) = 1$; 2) The existence of a horizon with a nonzero Hawking temperature requires that $f(z) > 0$ for $0 < z < z_h$, $f(z) = k(z_h - z)$ with $k > 0$ as $z \rightarrow z_h^-$ and there is no curvature singularity for $0 \leq z \leq z_h$. We further assume that both $w(z)$ and $f(z)$ are infinitely differentiable and nonvanishing outside the horizon, $0 < z < z_h$. Minimizing the Nambu-Goto action of the string world sheet embedded in the background metric (3.1), we find the parametric form of the function $\mathcal{A}(r)$

$$\begin{cases} r = 2\sqrt{F_c} \int_0^{z_c} \frac{dz}{\sqrt{f(F-F_c)}} \\ \mathcal{A} = \frac{\sqrt{\lambda}}{\pi} \left[\int_0^{z_c} dz \sqrt{\frac{F}{f}} \left(\frac{1}{\sqrt{1-\frac{F_c}{F}}} - 1 \right) - \int_{z_c}^{z_h} dz \frac{h}{z^2} \right] \end{cases} \quad (3.2)$$

where

$$F \equiv \frac{w^2(z)}{z^4} f(z) \quad (3.3)$$

and $F_c = F(z_c)$. The reality of the potential requires that z_c stays within the domain adjacent to the boundary where $F(z)$ is non-increasing. As $z_c \rightarrow 0$, we have $r \rightarrow 0$ and end up with the Coulomb like potential (2.8).

As a necessary condition for a smooth screening behavior, there must exists a z_c where $r \rightarrow \infty$ while \mathcal{A} stays finite. As we shall see that this is not the case. The integral (3.2) diverges for $z_c = z_h$ but the factor F_c removes the divergence of the limit $z_c \rightarrow z_h$. Consequently, the candidate potential \mathcal{A} will be restricted within a finite r like that of the super Yang-Mills, if the derivative of $F(z)$ is nonvanishing for $0 < z \leq z_h$. Alternatively, the integral may also diverge when $z_c < z_h$ but close to the point where the derivative of $F(z)$ vanishes. But in this case, both r of (3.2) and \mathcal{A} of (3.2) share the same divergence and a linear candidate potential at large distance emerges. A special example of the latter case, $f(z) = 1 - \frac{z^4}{z_h^4}$ and $w(z) = e^{\frac{1}{2}cz^2}$ was proposed in [33, 34] to generate the Cornell potential at low temperature².

To elaborate the above statements, let us examine the behavior near the horizon, where $F(z)$ vanishes according to the power law $F(z) \simeq K(z_h - z)^n$ with $K > 0$ and $n > 0$ (monotonically decreasing). It follows from (3.3) that the warp factor $w(z) \sim (z_h - z)^{\frac{n-1}{2}}$. For the super Yang-Mills, $n = 1$ and $K = \frac{4}{z_h^5}$. It is straightforward to calculate the Riemann tensor of the metric (3.1). Of particular interest is the component

$$R_{ijkl} = -e^{2\phi} f \phi'^2 (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \quad (3.4)$$

where $e^{2\phi} \equiv \frac{w(z)}{z^2}$ and the prime denotes the derivative with respect to z . It will contribute a term

$$12e^{-4\phi} f^2 \phi'^4 \quad (3.5)$$

to the invariant $R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda}$. It is straightforward to verify that this term diverges at $z = z_h$ for all $n > 0$ except that $n = 1$. This divergence will lead $R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda}$ to diverge because the contribution from all components are positive.

² At $T = 0$, the combination \mathcal{A} diverges because of the limit $z_h \rightarrow \infty$ of the integration of \mathcal{A}_1 . This reflects the nonexistence of an isolated heavy quark because of the color confinement. At a nonzero T , however, \mathcal{A} becomes finite and there is always a r_s where \mathcal{A} switch sign. In another word, the linearly confining potential is flattened out beyond r_s . There is no a clear cut distinction between a confined phase and a plasma phase in this regard. The deconfinement transition corresponds to a crossover from $r_s \sim \frac{T^3}{c^2} \exp\left(\frac{c}{2\pi^2 T^2}\right) \gg \frac{1}{T}$ at low temperature, $T \ll \sqrt{c}$, to $r_s \sim \frac{1}{T}$ at high temperature, $T \gg \sqrt{c}$.

Therefore we are left with only the case $n = 1$ to consider, which gives qualitatively the same behavior of the potential as the super Yang-Mills in the limit $z_c \rightarrow z_h$.

To isolate out the leading behavior as $z_c \rightarrow z_h$, We introduce δ such that $z_c - z_h \ll \delta \ll z_c$ and divide the integration domain $(0, z_c)$ into $(0, z_c - \delta)$ and $(z_c - \delta, z_c)$. In the latter domain, we may make the approximation $f(z) \simeq k(z_h - z)$ and $F(z) = K(z_h - z)$ with K another constant. Consequently

$$\int_{z_c - \delta}^{z_c} \frac{dz}{\sqrt{f(F - F_c)}} \simeq \frac{1}{\sqrt{kK}} \int_{f_c}^{k\delta} \frac{df}{\sqrt{f(f - f_c)}} \simeq \frac{1}{\sqrt{kK}} \ln \frac{k\delta}{f_c}. \quad (3.6)$$

In the former domain, we may set $F_c = 0$ in the integrand and end up with

$$\int_0^{z_c - \delta} \frac{dz}{\sqrt{f(F - F_c)}} \simeq \int_0^{z_h - \delta} \frac{dz}{\sqrt{fF}}, \quad (3.7)$$

which is independent of f_c . It follows that

$$r \simeq 2 \frac{\sqrt{f_c}}{k} \left(\ln \frac{1}{f_c} + \text{const.} \right) \quad (3.8)$$

as $z_c \rightarrow z_h$. Applying the same procedure to the first integral in (3.2), we obtain the leading behavior

$$\mathcal{A} \simeq \frac{\sqrt{\lambda} w_c}{\pi k z_c^2} f_c \left(\ln \frac{1}{f_c} + \text{const.} \right) \quad (3.9)$$

which returns to zero from the positive side. Because of the continuity of r and \mathcal{A} as functions of z_c , there exists a special value of z_c , $z_s \in (0, z_h)$, where the potential changes from attractive for $z_c < z_s$ to repulsive for $z_c > z_s$. If the distance r as a function of z_c consists of m local maxima outside the horizon, the candidate potential function $\mathcal{A}(r)$ will consists of $m + 1$ branches and we have $m \geq 1$ for the super Yang-Mills. The potential with three branches is shown in Fig.3. To reproduce the kink-like screening of Fig.2, the potential \mathcal{A} as a function of z_c has to be positive at the first local maximum of $r(z_c)$. The heavy quark potential will follow $\mathcal{A}(r)$ for $z < z_s$ and becomes flat afterwards.

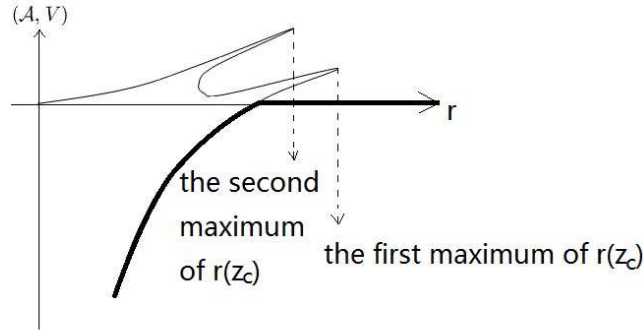


FIG. 3: An example of the candidate potential \mathcal{A} with three branches (thin line) together with the corresponding kink-like heavy quark potential V (thick line) of AdS/QCD.

Next, we explore the case where the derivative of $F(z)$ vanishes somewhere outside the horizon, i.e. $F'(z_0) = 0$ for $0 < z_0 < z_h$. As $z \rightarrow z_0$, we may approximate

$$F(z) \simeq a + b(z_0 - z)^n \quad (3.10)$$

with positive a and b , and an integer $n \geq 2$. For an even n , z_c has to stay on the side $z_c < z_0$ in order for the square root to be real. As $z_c \rightarrow z_0$, both the integral in (3.2) and the first integral for (3.2) diverges. Dividing the integration domain $(0, z_c)$ into $(0, z_c - \delta)$ and $(z_c - \delta, z_c)$ with $\epsilon \equiv |z_0 - z_c| \ll \delta \ll z_0$, we can easily extract the leading behaviors for $z_c \rightarrow z_0 - 0^+$,

$$\begin{cases} r \simeq \frac{2w_0}{\sqrt{b}z_0^2} \left(\ln \frac{z_0}{\epsilon} + \text{const.} \right) \\ \mathcal{A} \simeq \sqrt{\frac{\lambda}{bf_0}} \frac{a}{\pi} \left(\ln \frac{z_0}{\epsilon} + \text{const.} \right) \end{cases} \quad (3.11)$$

for $n = 2$ and

$$\begin{cases} r \simeq \frac{2}{n} B\left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right) \sqrt{\frac{a}{bf_0}} \epsilon^{1-\frac{n}{2}} \\ \mathcal{A} \simeq \frac{1}{n\pi} B\left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right) \sqrt{\frac{\lambda a^2}{bf_0}} \epsilon^{1-\frac{n}{2}} \end{cases} \quad (3.12)$$

for $n > 2$. Similar leading behavior as $z_c \rightarrow z_0 + 0^+$ read

$$\begin{cases} r \simeq \frac{2}{n} \left[B\left(\frac{1}{n}, \frac{1}{2}\right) + B\left(\frac{1}{n}, \frac{1}{2} - \frac{1}{n}\right) \right] \sqrt{abf_0} \epsilon^{1-\frac{n}{2}} \\ \mathcal{A} \simeq \frac{1}{n\pi} \left[B\left(\frac{1}{n}, \frac{1}{2}\right) + B\left(\frac{1}{n}, \frac{1}{2} - \frac{1}{n}\right) \right] \sqrt{\lambda a^2 bf_0} \epsilon^{1-\frac{n}{2}} \end{cases} \quad (3.13)$$

for an odd $n \geq 3$. In all cases above, the function $\mathcal{A}(r)$ contains a branch of Cornell potential. The examples of $F(z)$ and the corresponding $\mathcal{A}(r)$ and $V(r)$ for an even n and for an odd n are depicted in Figs. 4-5.

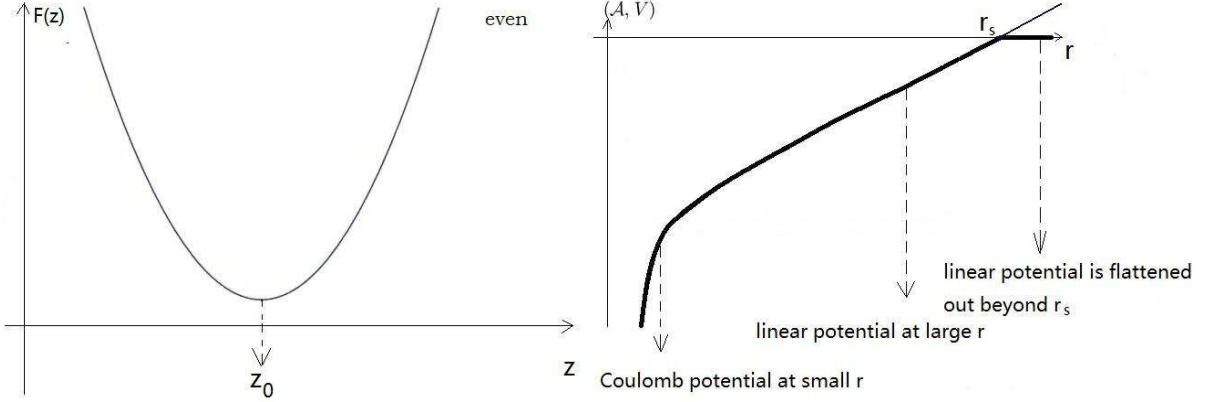


FIG. 4: The left panel shows the example of $F(z)$ whose expansion around z_0 is $F(z) \simeq a + b(z_0 - z)^n$ with an even n . We marked z_0 with dashed line. The right panel is the corresponding candidate potential \mathcal{A} as a function of r which is a Coulomb potential for small r combining with a linear potential at large r limit. The thick line represents the heavy quark potential V .

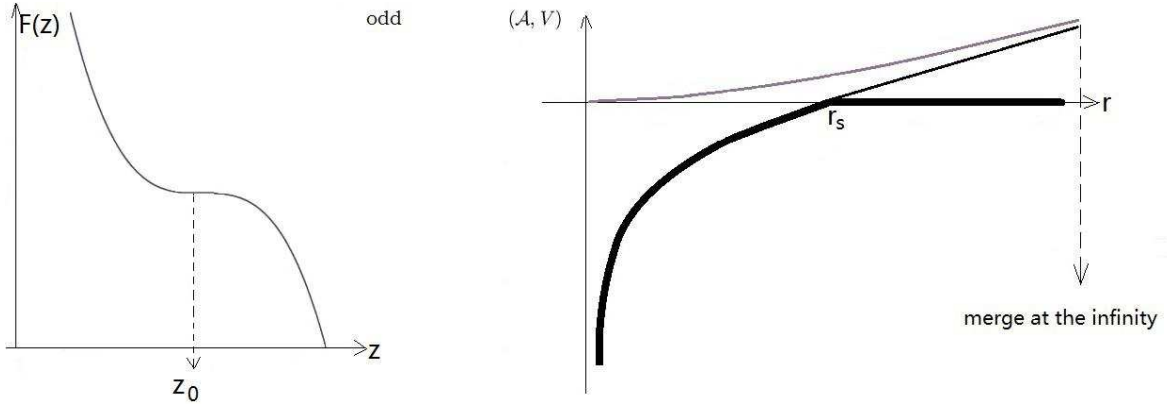


FIG. 5: The left panel shows the example of $F(z)$ whose expansion around z_0 is $F(z) \simeq a + b(z_0 - z)^n$ with an odd n . The right panel shows the two branches of the candidate potential \mathcal{A} as a function of r . Both branches rise linearly with r and will merge at $r \rightarrow \infty$. The thick line represents the heavy quark potential V .

4. DISCUSSIONS

In previous sections, we have explored all possible forms of the holographic heavy-quark potential at a nonzero temperature with a general metric that is asymptotically AdS and carries a black. The candidate potential function

$\mathcal{A}(r)$ is either supported within a finite range of the distance r (Figs.2-3) or takes the linear form for large r when r is allowed to go to infinity (Figs.4-5). The heavy quark potential $V(r)$ coincide with \mathcal{A} when the free energy of the an interacting quark-antiquark pair (represented by the world sheet of Fig.1a) is lower than that of a non-interacting pair (represented by the world sheet of Fig.1b) and is flattened out otherwise. A kink-like screening behavior with a discontinuity in the derivative dV/dr is developed then.

In this section we would like first to comment on some proposals in the literature to smooth out the kink-like the heavy quark potential in a thermal bath within the framework of the super Yang-Mills. The authors of [36] suggested that the parameter z_c becomes complex beyond the value when r is maximized. So the potential develops an imaginary part and decays with power law. This, however, cannot be the case at thermal equilibrium using the definition(2.4) since the thermal expectation value $\text{Tr} < \mathcal{P}(\vec{r})\mathcal{P}^\dagger(0) >$ is strictly real as is evident from the following reasoning:

$$\text{Tr} < \mathcal{P}(\vec{r})\mathcal{P}^\dagger(0) >^* = \text{Tr} < \mathcal{P}(0)\mathcal{P}^\dagger(\vec{r}) > = \text{Tr} < \mathcal{P}(-\vec{r})\mathcal{P}^\dagger(0) > = \text{Tr} < \mathcal{P}(\vec{r})\mathcal{P}^\dagger(0) > \quad (4.1)$$

where the second equality follows from the translation invariance and the last equality from the rotation invariance (the thermal expectation value should be a function of $|\vec{r}|$). One can obtain a complex potential using other definition or by analytic continuation[12, 29]. The authors of [37] suggested an alternative string world sheet by joining the two parallel world sheets of noninteracting quarks with a thin tube to represent the exchange of the lightest supergravity mode. While physically plausible, the area of such a configuration is always larger than that without the tube because the background metric (2.1) is *independent* of the transverse coordinates \vec{x} .

Next, we shall question the legitimacy of a kink-like screening potential from field theoretic point of view. Because of the $O(4)$ symmetry of the Lagrangian density, the path integral of a thermal relativistic field theory in R^3 at temperature T , is mathematically equivalent to the field theory under the same Lagrangian density but formulated in $R^2 \times S^1$ at zero temperature with a Euclidean time [37, 38], provide one of the spatial dimensions of the former, say x^3 is interpreted as the Euclidean time of the latter and the Euclidean time of the former is regarded as the compactified spatial dimension. The latter will be referred to as the adjoint field theory and is as well defined as the original thermal field theory. Consequently, a static Green's function, such as the heavy-quark potential of the thermal field theory, becomes a time-dependent Green's function of the adjoint field theory and the analyticity on the complex energy plane of its Fourier transformation should meet the requirements imposed by the unitarity and causality. Furthermore, the Fourier transformation should vanish at infinity of the energy plane in order for the Fourier integral of the retarded (advanced) Green's function exists. Let us examine if this is the case for a potential $V(r)$ that is continuous and vanishes for $r > r_s$ ³. The Fourier transformation

$$\mathcal{V}(q) = \int d^3\vec{r} e^{-i\vec{q}\cdot\vec{r}} V(r) = \frac{4\pi}{q} \int_0^{r_s} dr r V(r) \sin qr \equiv \mathcal{G}(iq_3, \vec{q}_\perp) \quad (4.2)$$

where $q = \sqrt{q_3^2 + \vec{q}_\perp^2} = \sqrt{-\omega^2 + \vec{q}_\perp^2}$ with $\omega = iq_3$. For a given \vec{q}_\perp and a sufficiently large real energy, we need to continue the integral to an imaginary q and this can be done at the integrand level because of the finite integration limits. We have

$$\mathcal{G}(\omega, \vec{q}_\perp) = \frac{4\pi}{\sqrt{\omega^2 - \vec{q}_\perp^2}} \int_0^{r_s} dr r V(r) \sinh \sqrt{\omega^2 - \vec{q}_\perp^2} r \quad (4.3)$$

$$= -\frac{4\pi}{\omega^2 - \vec{q}_\perp^2} \lim_{r \rightarrow 0} r V(r) - \frac{4\pi}{\omega^2 - \vec{q}_\perp^2} \int_0^{r_s} dr \frac{d}{dr} (r V) \cosh \sqrt{\omega^2 - \vec{q}_\perp^2} r \quad (4.4)$$

$$\simeq -\frac{2\pi}{\omega^3} e^{\omega r_s} r_s V'(r_s) \quad (4.5)$$

as $\omega \rightarrow \infty$, where an integration by part is conducted from the first line to the second line. A concrete example is the truncated Coulomb potential (2.9) and the corresponding

$$\mathcal{G}(\omega, \vec{q}_\perp) = \frac{4\pi\kappa}{\omega^2 - \vec{q}_\perp^2} [1 - i_0(\sqrt{\omega^2 - \vec{q}_\perp^2} r_s)] \quad (4.6)$$

with $i_0(z) = \sinh z/z$. While there are no complex singularities in ω -plane as is required by the unitarity, the exponentially growing behavior of $\mathcal{G}(\omega, \vec{q}_\perp)$ at large ω prevents us from defining the retarded and advance Green's

³ Here we assume that $\frac{dV}{dr} \neq 0$ at $r = r_s$ but this is not necessary for our conclusion. If the derivatives of V up to n -th order vanishes at r_s , we may conduct the integration by part n times more in the process of (4.5) below and the exponential factor $e^{\omega r_s}$ remains.

function of the adjoint field theory via Fourier integrals

$$G_{R(A)}(t, \vec{q}_\perp) \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \mathcal{G}(\omega \pm i0^+, \vec{q}_\perp) \quad (4.7)$$

with the upper(lower) sign for $R(A)$. We regard this problem a potential gap between the holographic approach and the field theoretic principle. It requires further investigations, perhaps some resummation of the finite N_c or and/or finite λ corrections to the leading form (2.2) to smear the kink of in the screening potential.

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- [1] T. Matsui, H. Satz, *J/Ψ suppression by quark-gluon plasma formation*, Phys. Lett. **B178**, (1986) 416-422.
 - [2] E. Eichten, K. Gottfried, T. Konoshita, K. D. Lane, and T. -M. Yan, *Charmonium: The model*, Phys. Rev. **D17**, (1978) 3090-3117.
 - [3] E. Eichten, and F. Feinberg, *Spin-dependent forces in quantum chromodynamics*, Phys. Rev. **D23**, (1981) 2724-2744.
 - [4] John D. Stack, *Heavy-quark potential in SU(3) lattice gauge theory*, Phys. Rev. **D29**, (1984) 1213-1218.
 - [5] F. Karsch, and H.W. Wyld, *Screening of the QCD heavy quark potential at finite temperature*, Phys. Lett. **B213**, 4 (1988) 505-510.
 - [6] U.M. Heller, Khalil M. Bitar, R.G. Edwards, and A.D. Kennedy, *The heavy quark potential in QCD with 2 flavors of dynamical quarks*, Phys. Lett. **B335**, 1 (1994) 71-76 [hep-lat/9401025].
 - [7] O. Kaczmarek, F. Karsch, E. Laermann, and M. Lutgemeier, *Heavy quark potentials in quenched QCD at high temperature*, Phys. Rev. **D62** (2000) 034021 [hep-lat/9908010].
 - [8] O. Kaczmarek, F. Karsch, F. Zantow and P. Petreczky, *Static quark-antiquark free energy and the running coupling at finite temperature*, Phys. Rev. **D70**, (2004) 074505
 - [9] O. Kaczmarek, F. Karsch, P. Petreczkyb, and F. Zantow, *Heavy quark free energies, potentials and the renormalized Polyakov loop*, Nucl. Phys. **B**, 129-130 (2004) 560-562 [hep-lat/0309121].
 - [10] S. Digal, O. Kaczmarek, F. Karsch, and H. Satz, *Heavy quark interactions in finite temperature QCD*, Eur. Phys. J. **C43**, 71-75 (2005) [hep-ph/0505193].
 - [11] H. Satz, *Colour deconfinement and quarkonium binding*, J. Phys. **G32**, R25 (2006) [hep-ph/0512217].
 - [12] Alexander Rothkopf, Tetsuo Hatsuda, and Shoichi Sasaki, *Complex Heavy-Quark Potential at Finite Temperature from Lattice QCD*, Phys. Rev. Lett. **108**, (2012) 162001 [arXiv: 1108.1579 [hep-lat]].
 - [13] H. A. Weldon, *Covariant calculations at finite temperature: The relativistic plasma*, Phys. Rev. **D26**, (1982) 1394-1407.
 - [14] R. D. Pisarski, *Scattering Amplitudes in Hot Gauge Theories*, Phys. Rev. Lett. **63**, (1989) 1129-1132.
 - [15] G. t. Hooft, *Dimensional reduction in quantum gravity*, [gr-qc/9310026].
 - [16] L. Susskind, *The world as a hologram*, J. Math. Phys. **36** :6377-6396 (1995) [hep-th/9409089].
 - [17] Juan Maldacena, *Wilson Loops in Large N Field Theories*, Phys. Rev. Lett. **80**, (1998) 4859-4863 [hep-th/9803002].
 - [18] A. Brandhuber, N. Itzhaki, J. Sonnenschein, and S. Yankielowicz, *Wilson loops in the large N limit at finite temperature*, Phys. Lett. **B434**, (1998) 36-40 [hep-th/9803137].
 - [19] S. J. Rey, S. Theisen and J. T. Yee, *Wilson-Polyakov loop at finite temperature in large-N gauge theory and anti-de Sitter supergravity*, Nucl. Phys. **B527**, (1998) 171-186 [hep-th/9803135].
 - [20] Shao-xia Chu, Defu Hou, and Hai-cang Ren, *The subleading term of the strong coupling expansion of the heavy-quark potential in a N = 4 super Yang-Mills vacuum*, JHEP **08**, (2009) 004 [arXiv: 0905.1874 [hep-ph]].
 - [21] V. Foroni, *Quark-antiquark potential in AdS at one loop*, JHEP **11**, (2010) 079 [arXiv: 1009.3939 [hep-th]].
 - [22] Zi-qiang Zhang, Defu Hou, Hai-cang Ren, and Lei Yin, *The subleading term of the strong coupling expansion of the heavy-quark potential in a N = 4 super Yang-Mills plasma*, JHEP **07**, (2011) 035 [arXiv: 1104.1344 [hep-ph]].
 - [23] Danning Li, Mei Huang, Qi-Shu Yan, *A dynamical holographic QCD model for chiral symmetry breaking and linear confinement*, Eur. Phys. J. **C73** (2013) 2615 [arXiv: 1206.2824 [hep-th]].
 - [24] Rong-Gen Cai, Song He, Danning Li, *A hQCD model and its phase diagram in Einstein-Maxwell-Dilaton system*, JHEP **03**, (2012) 033 [arXiv: 1201.0820 [hep-th]].
 - [25] Danning Li, Mei Huang, *Dynamical holographic QCD model for glueball and light meson spectra*, [arXiv: 1303.6929 [hep-ph]].
 - [26] Jorge Noronha, Miklos Gyulassy, and Giorgio Torrieri, *Conformal holography of bulk elliptic flow and heavy-quark quenching in relativistic heavy ion collisions*, Phys. Rev. **C 82**, (2010) 054903 [arXiv: 1009.2286 [nucl-th]].
 - [27] Mohammed Mia, Keshav Dasgupta, Charles Gale, and Sangyong Jeon, *Toward large N thermal QCD from dual gravity: The heavy quarkonium potential*, Phys. Rev. **D 82**, (2010) 026004 [arXiv: 1004.0387 [hep-th]].

- [28] Mohammed Mia, Keshav Dasgupta, Charles Gale, and Sangyong Jeon, *Heavy quarkonium melting in large N thermal QCD*, Phys. Lett. **B694**, (2011) 460-466 [arXiv: 1006.0055 [hep-th]].
- [29] Tomoya Hayata, Kanabu Nawa, Tetsuo Hatsuda, *Time-dependent heavy-quark potential at finite temperature from gauge-gravity duality*, Phys.Rev. **D87**, (2013) 101901(R) [arXiv: 1211.4942 [hep-ph]].
- [30] Willy Fischler, Sandipan Kundu, *Strongly coupled gauge theories: high and low temperature behavior of non-local observables*, JHEP **05** (2013) 098 [arXiv: 1212.2643 [hep-th]].
- [31] Yan Wu, Defu Hou and Hai-cang Ren, *The relativistic correction of the quarkonium melting temperature with a holographic potential*, Phys. Rev. **C87**, (2013) 025203 [arXiv: 1210.6748 [hep-ph]].
- [32] A. Karch, E. Katz, D. T. Son and M. A. Stephanov, *Linear confinement and AdS/QCD*, Phys. Rev. **D74**, (2006) 015005 [hep-ph/0602229].
- [33] Oleg Andreev, Valentin I. Zakharov, *Heavy-Quark Potentials and AdS/QCD*, Phys. Rev. **D74**, (2006) 025023 [hep-ph/0604204].
- [34] Oleg Andreev, Valentin I. Zakharov, *On Heavy-Quark Free Energies, Entropies, Polyakov Loop, and AdS/QCD*, JHEP **04** (2007) 100 [hep-ph/0611304].
- [35] J P Shock, F. Wu, Y. L. Wu and Z. F. Xie, *AdS/QCD phenomenological models from a back-reacted geometry*, JHEP **03** (2007) 064 [hep-ph/0611227].
- [36] J. L. Albacete, Y. V. Kovchegov and A. Taliotis, *Heavy-quark potential at finite temperature using the holographic correspondence*, Phys. Rev. **D78**, (2008) 115007 [arXiv: 0807.4747 [hep-th]].
- [37] D. Bak, A. Karch and L. Yaffe, *Debye screening in strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma*, JHEP **08** (2007) 049 [arXiv: 0705.0994 [hep-th]].
- [38] De-fu Hou, Jia-rong Li, Hui Liu, and Hai-cang Ren, *The momentum analyticity of two point correlators from perturbation theory and AdS/CFT*, JHEP **07** (2010) 042 [arXiv: 1003.5462 [hep-ph]].